Senior Mathematical Challenge

Tuesday 3 October 2023

For reasons of space, these solutions are necessarily brief.

There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation.

There is also a version of this document available on the UKMT website which includes each of the questions alongside its solution:

www.ukmt.org.uk

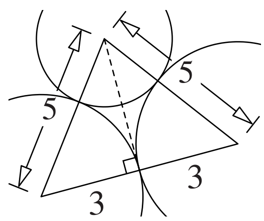
1. **C** The prime factorisation of 2023 is $7 \times 17 \times 17$ so $\sqrt{\dfrac{2023}{2+0+2+3}} = \sqrt{\dfrac{2023}{7}} = \sqrt{17^2} = 17$.

2. **C** The difference between one third and 0.333 is $\dfrac{1}{3} - \dfrac{333}{1000} = \dfrac{1000-999}{3000} = \dfrac{1}{3000}$.

3. **D** The new area = the old area $\times 1.2 \times 0.85$ = the old area $\times 1.02$. This represents a 2% increase.

4. **C** The world record of 5000 m in 19 minutes and 6 seconds $\simeq$ 5000 m in 20 minutes = 15000 m in 60 minutes = 15000 m in an hour = 15 km/h.

5. **B** The triangle formed by joining the centres of the circles is isosceles, so splitting it along its line of symmetry gives us two right-angled triangles each with a base of 3 and a hypotenuse of 5. Using Pythagoras' Theorem the perpendicular height is 4. The area of the whole triangle is then $\frac{1}{2} \times 6 \times 4 = 12$.



6. **B** The sum of any three integers in arithmetic progression is a multiple of 3. For proof of this, if we let the smallest integer be a and the common difference of the sequence be d, then $a + (a + d) + (a + 2d) = 3a + 3d = 3(a + d)$. As a result of the way the grid is filled, all the horizontal, vertical and diagonal lines contain numbers which are in arithmetic progression. Horizontally there are 2 lines of three cells in each of the 4 rows. Here $d = 1$. Vertically, there are again 2 lines in each of the 4 columns. Here $d = 4$. On the diagonals with positive gradient, there are 4 lines, with $d = -3$. On the diagonals with negative gradient there are four lines with $d = 5$. This is a total of $8 + 8 + 4 + 4 = 24$ lines.

7. **D** The sequence begins 2023, 2022, 1, 2021, 2020, 1, 2019, 2018, 1 ... . Let the k^{th} term be u_k. Now consider the sequence $u_1, u_4, u_7, \ldots$, which starts 2023, 2021, 2019, ... Here the terms decrease by two each time. Since $25 = 1 + 8 \times 3$, $u_{25} = u_1 - 8 \times 2 = 2023 - 16 = 2007$.

8. **A** The value of $99(0.\dot{4}\dot{9} - 0.\dot{4}) = 99\left(\dfrac{49}{99} - \dfrac{4}{9}\right) = 99\left(\dfrac{49}{99} - \dfrac{44}{99}\right) = 99\left(\dfrac{49-44}{99}\right) = 99 \times \dfrac{5}{99} = 5$.

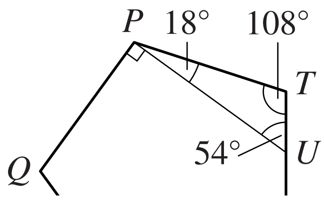
9. D For 1 Down, $2 \times 2^5 = 64$ is too small and $2 \times 4^5 = 2048$ is too big and therefore we must have $2 \times 3^5 = 486$. 3 Across must then start with a 6 and is therefore $5^4 = 625$. 2 Down must then end in a 5 and is therefore $5^3 = 125$. 1 Across is then 4 * 1. The only square of this form is $21^2 = 441$, so * is a 4.

10. B The sum $2^{2024} + 2^{2023} + 2^{2022}$ can be factorised to $2^{2022}(2^2 + 2^1 + 1) = 2^{2022} \times 7$. Hence, of the numbers listed, only 7 and $8 = 2^3$ are factors of 2^{2022}.

11. B Each of the four people is either telling the truth or lying. Assume first that Wenlu is telling the truth, then Xander is lying, which implies that Yasser is telling the truth which finally implies that Zoe is also telling the truth. In this case 3 people tell the truth. Now assume that Wenlu is lying. Therefore Xander is telling the truth that Yasser is lying and finally Zoe is also lying. In this case only 1 person tells the truth. In both cases, all four statements are consistent with each other.

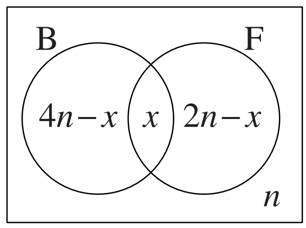
12. E As $n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n$, factors of 50! which contain a factor of 7 are 7, 14, 21, 28, 35, 42 and 49. The first six of these each contribute a single factor of 7 and 49 contributes two. The greatest power of 7 which is a factor of 50! is then 7^8, so $k = 8$.

13. A The interior angles of a regular pentagon are $180° - \frac{360°}{5} = 108°$. As $\angle QPU$ is a right angle, $\angle UPT = 108° - 90° = 18°$. As angles in a triangle sum to $180°$, $\angle PUT = 180° - (108° + 18°) = 54°$. Therefore $\angle TPU : \angle PUT : \angle UTP = 18 : 54 : 108 = 1 : 3 : 6$.

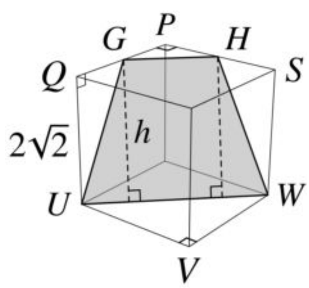


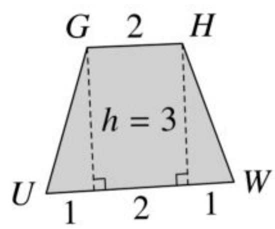
14. E The equation of the circle is $x^2 + y^2 = r^2$. At Q, $(12 - d)^2 + (2d - 6)^2 = r^2$. At P, $d^2 + (-d)^2 = r^2$, so $2d^2 = r^2$. Expanding the first equation and subtracting the second gives $144 - 24d + d^2 + 4d^2 - 24d + 36 - 2d^2 = 0$, which simplifies to $3d^2 - 48d + 180 = 0$. Dividing by 3 and factorising gives $(d - 6)(d - 10) = 0$. Therefore $d = 6$ or $d = 10$ and the sum of these values is 16.

15. D Let the number of people who play both basketball and football be x and the number who play neither be n. A Venn diagram can then be filled as shown. As there are 30 students, $7n - x = 30$. As $x \geq 0$, $7n - 30 \geq 0$ and so $n \geq 5$. From the Venn diagram it can be seen that $2n - x \geq 0$, therefore $2n - (7n - 30) \geq 0$ so $n \leq 6$. So $n = 5$ or 6 and the corresponding values of x are 5 or 12. The only one of these in the listed options is $x = 5$.



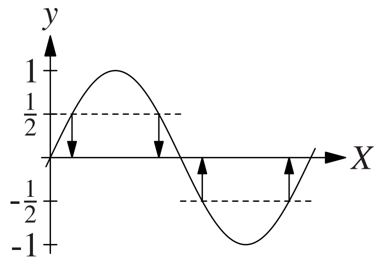
16. A To find the area of the trapezium, we require lengths of GH, UW and the perpendicular distance between them, h, say. In triangle PGH, $PG = PH = \sqrt{2}$ therefore $GH = \sqrt{\sqrt{2}^2 + \sqrt{2}^2} = 2$.
Triangle VUW is an enlargement of triangle PGH with scale factor 2, so $UW = 4$. In order to find h we must first find length GU. In triangle QUG, $UG = \sqrt{(2\sqrt{2})^2 + \sqrt{2}^2} = \sqrt{10}$.
From the triangular end of the trapezium, it follows that $1^2 + h^2 = \sqrt{10}^2$ therefore $h = 3$. The area of the trapezium $= \frac{1}{2} \times (4 + 2) \times 3 = 9$.





17. **E** In order to get the largest number, N, we need to make its earlier digits as large as possible, starting 9876 . . . as far as this works. However, since 53, 43, 23 and 13 are all prime, the digit 3 must precede all of 5, 4, 2 and 1. So the latest 3 can come is immediately after 6. Thereafter there are no reasons not to follow numerical order, making $N = 987635421$. Its 5^{th} and 6^{th} digits are 3 and 5.

18. **C** The equation $1 + 2\sin X - 4\sin^2 X - 8\sin^3 X = 0$ factorises to give $(1 + 2\sin X) - 4\sin^2 X(1 + 2\sin X) = 0$ and then to $(1 + 2\sin X)(1 - 4\sin^2 X) = 0$. Fully factorised, we have $(1 + 2\sin X)(1 + 2\sin X)(1 - 2\sin X) = 0$. So $\sin X = -\frac{1}{2}$ or $\sin X = \frac{1}{2}$. For $0° < X < 360°$, there are then four solutions as shown in the diagram.



19. **E** The expression $\dfrac{7n + 12}{2n + 3} \equiv \dfrac{4(2n + 3)}{2n + 3} - \dfrac{n}{2n + 3} \equiv 4 - \dfrac{n}{2n + 3}$. The first expression takes integer values precisely when $\dfrac{n}{2n + 3}$ is an integer.

Consider first $n > 0$. When $n > 0$, $2n + 3 > n$, therefore $\dfrac{n}{2n + 3} < 1$ so no integer values of the expression are possible.

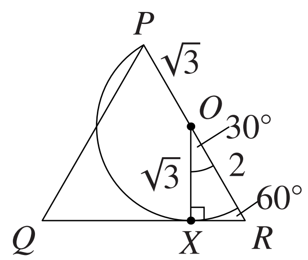
Next, consider $n = 0$. In this case, $\dfrac{n}{2n + 3} = \dfrac{0}{0 + 3} = 0$ which is an integer.

When $n < 0$, in order to form an integer, we require $n \leq 2n + 3$, therefore $n \geq -3$.

Possible values of n are then $n = -1$, -2 and -3. The values of $\dfrac{n}{2n + 3}$ in these cases are $\dfrac{-1}{2 \times (-1) + 3} = -1$, $\dfrac{-2}{2 \times (-2) + 3} = 2$ and $\dfrac{-3}{2 \times (-3) + 3} = 1$. Therefore the sum of the integer values of the initial expression is $(4 - 0) + (4 - (-1)) + (4 - 2) + (4 - 1) = 14$.

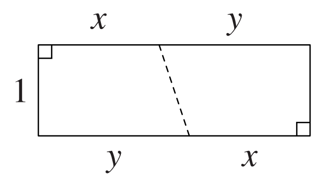
20. **C** Using Pythagoras' Theorem, $NM = \sqrt{(p + q)^2 + r^2}$ so the two journeys have lengths $p + r$ and $q + \sqrt{(p + q)^2 + r^2}$. Equating and rearranging, $p + r - q = \sqrt{(p + q)^2 + r^2}$ and so $(p + r - q)^2 = (p + q)^2 + r^2$. Expanding leads to $p^2 + 2pr - 2pq + r^2 - 2qr + q^2 = p^2 + 2pq + q^2 + r^2$ and therefore $2pr - 2qr = 4pq$. Rearranging to give q in terms of p and r, $pr = q(2p + r)$ so $q = \dfrac{pr}{2p + r}$.

21. **D** As the semicircle touches QR at X, the radius OX and tangent QR are perpendicular as shown. Triangle OXR is a 30°, 90°, 60° triangle and OX is given as $\sqrt{3}$. Therefore $XR = 1$ and $OR = 2$. As OP is also a radius of the circle, $OP = \sqrt{3}$ and $PR = QR = 2 + \sqrt{3}$. The length $QX = (2 + \sqrt{3}) - 1 = 1 + \sqrt{3}$.

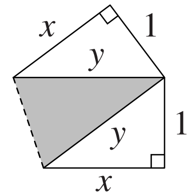


22. **C** Let $z = (\cos^{-1} x)$. Then $x = \cos z$ and $y = \sin z$ and therefore $x^2 + y^2 = 1$. As z lies between 0° and 180°, x lies between -1 and 1 and y lies between 0 and 1. Hence we get the upper semicircle shown on the graph in option C.

23. D In the first diagram shown, the paper is to be folded so that the bottom left vertex will lie on top of the top right vertex in order to form the desired pentagon. The fold line, shown dotted, must therefore lie on the perpendicular bisector of the line joining the bottom left and top right vertices and so pass through the centre of the rectangle. Labelling the longest sides of the rectangle with x and y,



$$x + y = 3. \qquad (1)$$



From the folded diagram, we have two right-angled triangles and in each, $1 + x^2 = y^2$. Rearranging and factorising gives

$$1 = (y + x)(y - x). \qquad (2)$$

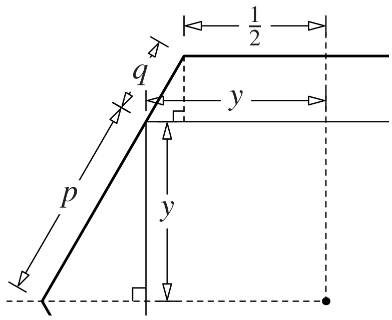
Substituting (1) into (2) gives $1 = 3(y - x)$ and so

$$\tfrac{1}{3} = y - x. \qquad (3)$$

Solving (1) and (3) leads to $y = \frac{5}{3}$ and $x = \frac{4}{3}$. The area of the pentagon = 1×3 – the shaded area. As the shaded area can be viewed as a triangle with base y and therefore perpendicular height 1, the area of the pentagon $= 3 - \frac{1}{2} \times y \times 1 = 3 - \frac{1}{2} \times \frac{5}{3} \times 1 = \frac{13}{6}$.

The area of the pentagon as a fraction of the area of the original rectangle is $\dfrac{\frac{13}{6}}{3} = \dfrac{13}{18}$.

24. B Let the square have side-length $2y$.
The two triangles shown, with hypotenuses p and q, have angles $30°$, $60°$ and $90°$. As the hexagon has side-length 1,



$$p + q = 1. \qquad (1)$$

From the larger triangle and from the top left of the square,

$$y = \frac{\sqrt{3}p}{2} \qquad \text{and} \qquad y = \tfrac{1}{2}q + \tfrac{1}{2}. \qquad (2)$$

Equating the two equations in (2) and rearranging gives $\sqrt{3}p - q = 1$. (3)
Solving (1) and (3) simultaneously gives $(\sqrt{3} + 1)p = 2$.
Rearranging and rationalising leads to

$$p = \frac{2}{(\sqrt{3} + 1)} \frac{(\sqrt{3} - 1)}{(\sqrt{3} - 1)} = \sqrt{3} - 1.$$

Therefore, the length of the side of the square $2y = \sqrt{3}(\sqrt{3} - 1) = 3 - \sqrt{3}$.

25. A Factorising $x^3y^2 - x^2y^2 - xy^4 + xy^3 \geq 0$ gives $xy^2(x^2 - x - y^2 + y) \geq 0$. Rearranging to $xy^2(y - x - (y^2 - x^2)) \geq 0$ and then factorising gives $xy^2(y - x)(1 - y - x) \geq 0$. As $0 \leq x \leq y$, we know that $x \geq 0$, $y^2 \geq 0$ and $(y - x) \geq 0$ so the fourth factor, $(1 - y - x) \geq 0$. This rearranges to $y \leq 1 - x$. The lines $y = x$ and $y = 1 - x$ meet at $(\frac{1}{2}, \frac{1}{2})$ so the shaded region has area $\frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{4}$.